Topic 3 –
Some Topology

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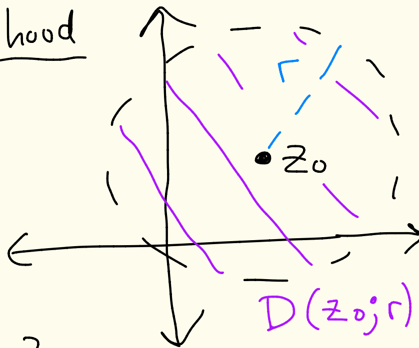
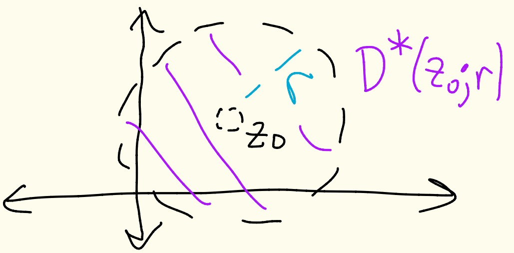
Def: Let $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}$ with $r > 0$.

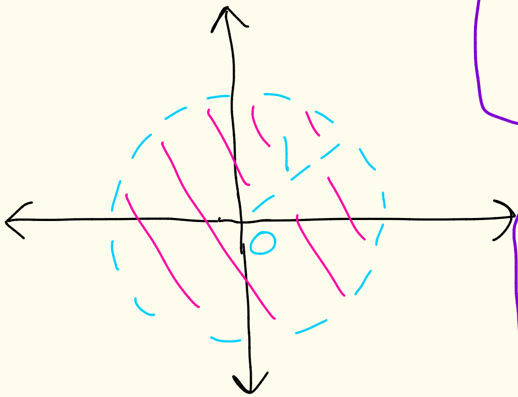
The set
$$D(z_0; r) = \{ z \in \mathbb{C} \mid |z - z_0| < r \}$$
is called an <u>r-neighborhood of z_0</u>.



The set
$$D^*(z_0; r) = D(z_0; r) - \{z_0\}$$
$$= \{ z \in \mathbb{C} \mid 0 < |z - z_0| < r \}$$
is called a <u>deleted r-neighborhood of z_0</u>.



Ex: $D(0;1)$



Ex: $\frac{i}{2} \in D(0;1)$

Since $\left|\frac{i}{2} - 0\right| = \sqrt{\left(\frac{1}{2}\right)^2}$
$= \frac{1}{2} < 1$

Ex: $1+i \notin D(0;1)$

Since $|1+i-0| = \sqrt{1^2+1^2}$
$= \sqrt{2} \approx 1.4 > 1$

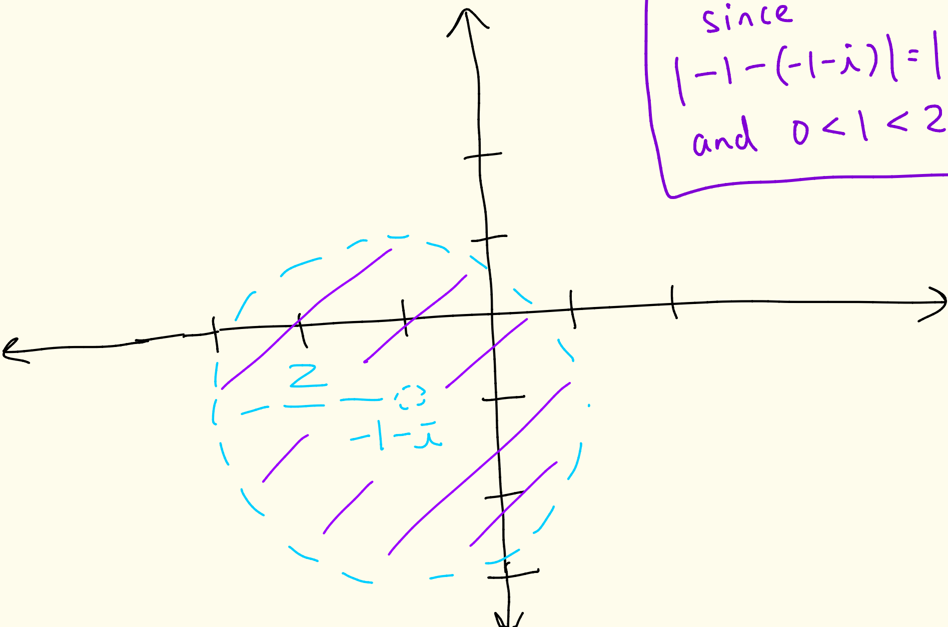
Ex: $D^*(-1-i\,;\,2)$

Ex:
$-1 \in D^*(-1-i;2)$

since

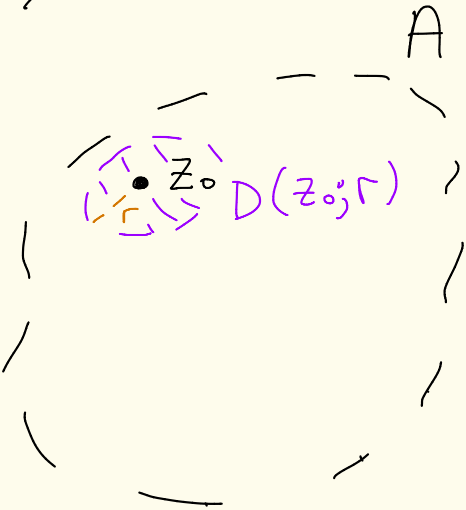
$|-1-(-1-i)| = |i| = 1$

and $0 < 1 < 2$.

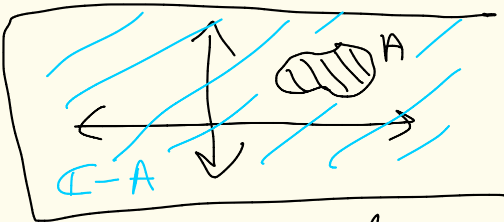


Def: Let $A \subseteq \mathbb{C}$.

(1) We say that $z_0 \in A$ is an **interior point of A** if there exists $r > 0$ where $D(z_0; r) \subseteq A$.



(2) We say that A is an **open set** if every point in A is an interior point of A.



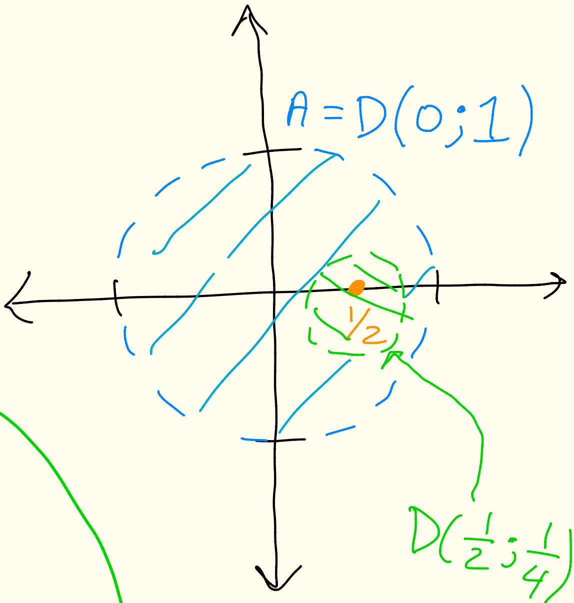
(3) We say that A is **closed** if its complement $\mathbb{C} - A$ is open.

center radius

Ex: Let

$$A = D(0;1).$$

Show that $\frac{1}{2}$ is an interior point of A.



Claim: $D(\frac{1}{2};\frac{1}{4}) \subseteq A$

proof: Let $w \in D(\frac{1}{2};\frac{1}{4})$

Thus, $|w - \frac{1}{2}| < \frac{1}{4}$.

To show that $w \in A$ we must show that $|w-0|<1$.

We have that

$$\begin{aligned} |w-0| &= |w - \tfrac{1}{2} + \tfrac{1}{2}| \\ &\leq |w-\tfrac{1}{2}| + |\tfrac{1}{2}| \\ &< \tfrac{1}{4} + \tfrac{1}{2} \\ &= \tfrac{3}{4} < 1. \end{aligned}$$

Thus, $w \in A$.

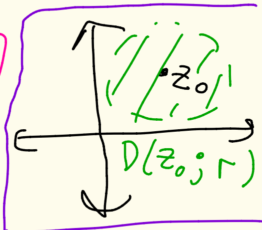
So, $D(\frac{1}{2};\frac{1}{4}) \subseteq A$. $\blacksquare$

By the claim, $\frac{1}{2}$ is an interior point of A.

Theorem: Let $z_0 \in \mathbb{C}$, $r \in \mathbb{R}$, $r > 0$.

Then

① $D(z_0; r) = \{ z \mid |z - z_0| < r \}$ is open

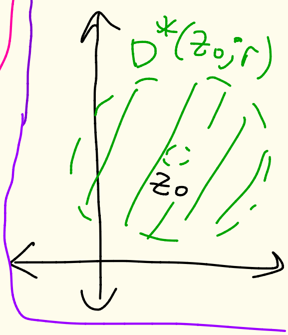


and

② $D^*(z_0; r) = D(z_0; r) - \{z_0\}$

$= \{ z \mid 0 < |z - z_0| < r \}$

is open.



pf: We prove ① in class.

See below for part ②

proof of ①:

Let $z \in D(z_0; r)$.

Let $\varepsilon = r - |z - z_0|$

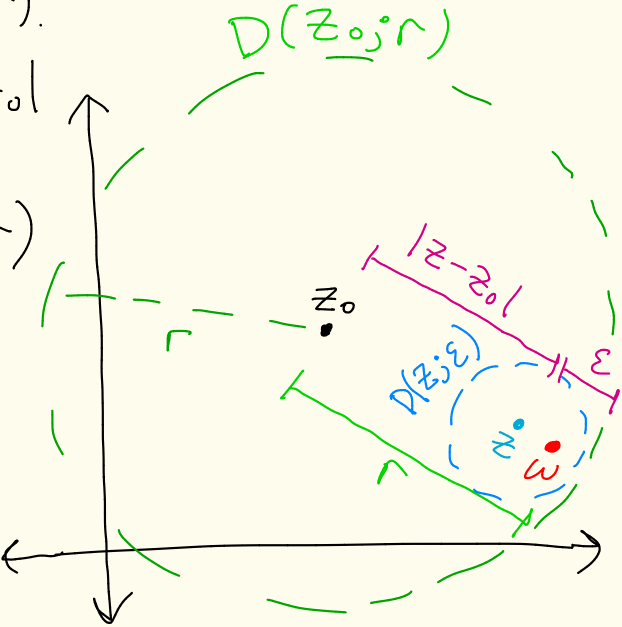
We now show

$D(z; \varepsilon) \subseteq D(z_0; r)$

Let $w \in D(z; \varepsilon)$

So,

$|w - z| < \varepsilon$.



Then,

$$\begin{aligned}
|w - z_0| &= |w - z + z - z_0| \\
&\leq |w - z| + |z - z_0| \\
&< \varepsilon + |z - z_0| \\
&= r - |z - z_0| + |z - z_0| \\
&= r.
\end{aligned}$$

So, $|w - z_0| < r$.

Thus, $w \in D(z_0; r)$.

Therefore $D(z; \varepsilon) \subseteq D(z_0; r)$.

So, z is an interior point of $D(z_0; r)$.

Thus, $D(z_0; r)$ is open.

$\boxed{1}$

$\boxed{\text{proof of ②}}$:

Let's show that $D^*(z_0;r)$ is open.

Let $z \in D^*(z_0;r)$.

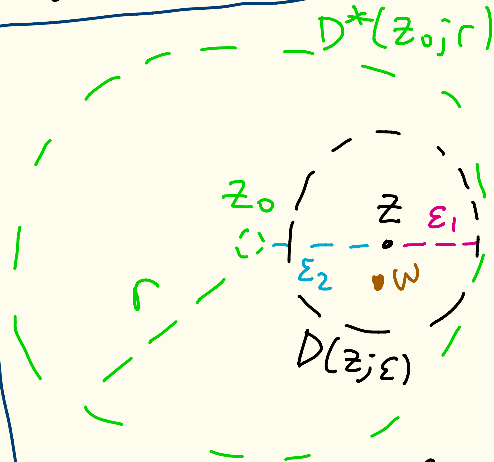
Note $z \neq z_0$.

We must show that z is an interior point of $D^*(z_0;r)$

Let $\varepsilon_1 = r - |z - z_0|$ as in the proof of part ①

Let $\varepsilon_2 = |z - z_0|$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.



This pic is $\varepsilon_1 < \varepsilon_2$
So $\varepsilon = \varepsilon_1$

<u>Claim:</u> $D(z;\varepsilon) \subseteq D^*(z_0;r)$

this will show that z is an int. point of $D^*(z_0;r)$ and complete the proof of part ②

Let $w \in D(z;\varepsilon)$.

We must show that $0 < |w - z_0| < r$.

Why must $0 < |w - z_0|$?

Suppose $0 = |w - z_0|$

Then $w = z_0$.

But then

$$|w-z| = |z_0 - z| = \varepsilon_2 \geq \varepsilon$$

But $w \in D(z;\varepsilon)$ so $|w-z| < \varepsilon$.

Contradiction.

Thus, $w \neq z_0$ and so $0 < |w-z_0|$.

Why must $|w-z_0| < r$?

We have that

$$\begin{aligned} |w-z_0| &= |w-z+z-z_0| \\ &\leq |w-z| + |z-z_0| \\ &< \varepsilon + |z-z_0| \\ &\leq \varepsilon_1 + |z-z_0| \\ &= (r-|z-z_0|) + |z-z_0| \\ &= r \end{aligned}$$

So, $|w-z_0| < r$.

Hence from above, we have $0 < |w-z_0| < r$

So, $D(z;\varepsilon) \subseteq D^*(z_0;r)$.

Thus, z is an int point and $D^*(z_0;r)$ is open $\blacksquare$

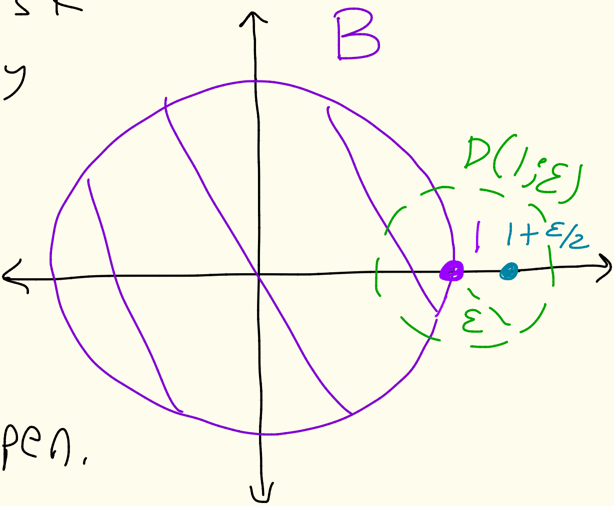
$\boxed{\text{Ex:}}$ Show that
$$B = \{ z \mid |z| \le 1 \}$$
is not open.

Note that $1 \in B$.
We will show that 1 is not an interior point of B.

Suppose we pick some $\varepsilon > 0$ and look at $D(1;\varepsilon)$.

Note that $1 + \frac{\varepsilon}{2} \in D(1;\varepsilon)$
but $1 + \frac{\varepsilon}{2} \notin B$.

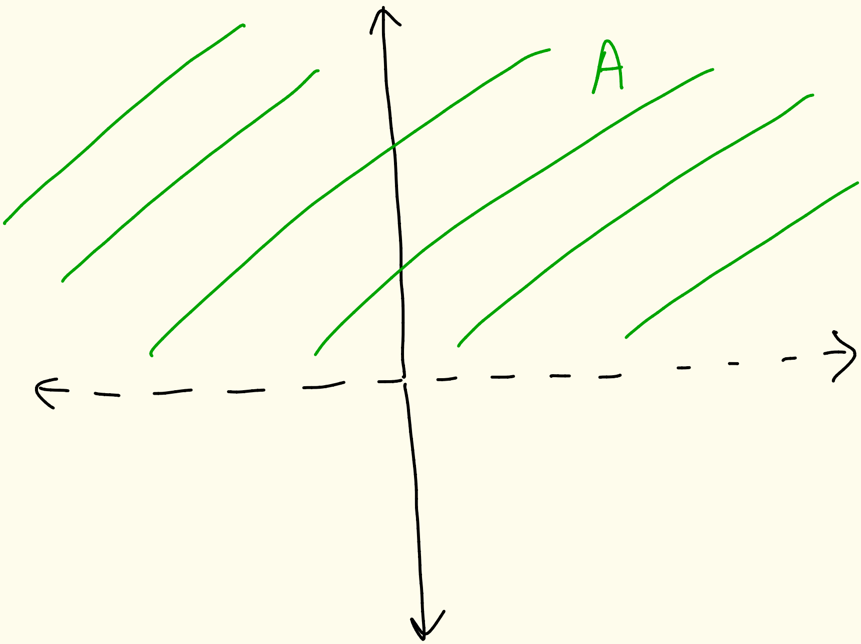
There is not disk $D(1;\varepsilon)$ completely contained in B.



So, $1 \in B$ but not an interior point of B.

So, B is not open.

Ex: $A = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$

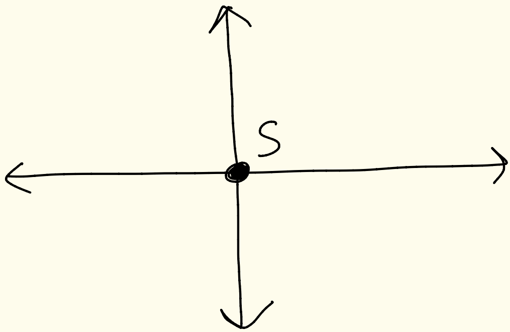


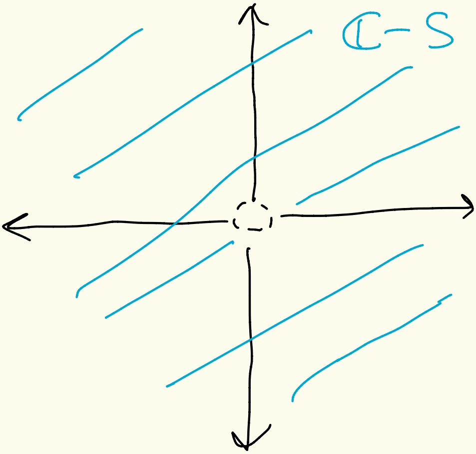
A is open.

See HW 3 problem 2(c)

<u>Recall:</u> $S \subseteq \mathbb{C}$ is <u>closed</u> if $\mathbb{C} - S$ is open.

<u>Ex:</u> $S = \{0\}$





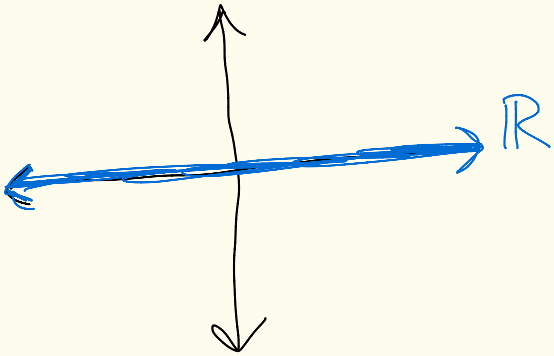
$\mathbb{C} - S$ is open

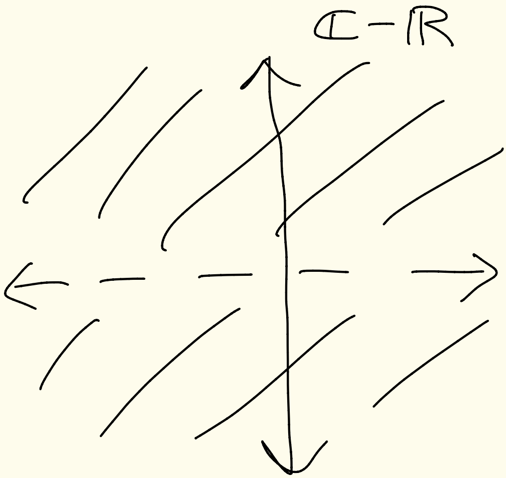
So, S is closed.

$\boxed{\text{See HW 3 problem 3(e)}}$

Also, S is not open.

$\boxed{\text{Ex:}}$ $S = \mathbb{R}$

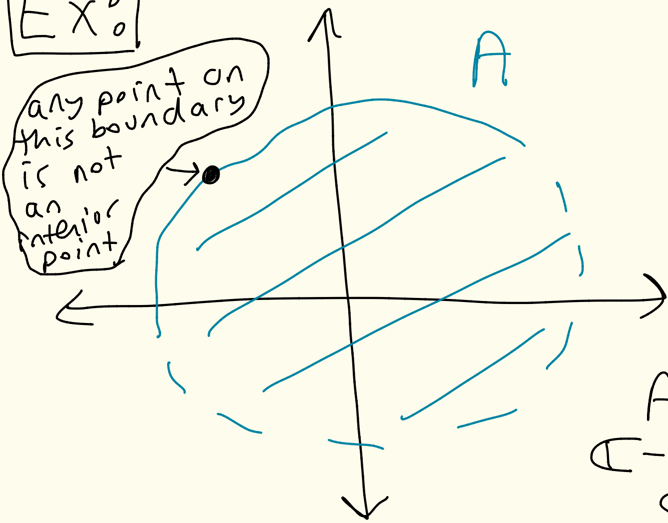


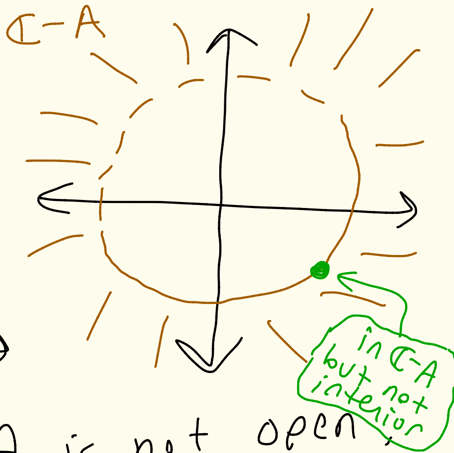


$\mathbb{R}$ is not open.

$\mathbb{C} - \mathbb{R}$ is open, so $\mathbb{R}$ is closed.

$\boxed{\text{Ex:}}$





A is not open,

$\mathbb{C} - A$ is not open,

So, A is not closed

Thm: Let $A, B \subseteq \mathbb{C}$.

Then:

1. $\emptyset$ is open and closed.
2. $\mathbb{C}$ is open and closed.
3. If A is open and B is open then $A \cup B$ is open and $A \cap B$ is open.
4. If A is closed and B is closed then $A \cup B$ is closed and $A \cap B$ is closed.

pf: See HW 3. $\blacksquare$